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On the Polystability of Dynamical Systems

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1. INTRODUCTION

Stability analysis of nonlinear systems [4, 5] is made under the assumption of the "equality" of all the solution coordinates with respect to the dynamical properties as accepted in the classical papers by Lyapunov and his adherents (see, for example, [3]). The exception is made for stability with respect to a part of the variables (see [8]) which is divided into two subvectors, the norm of one of which is to be "nonincreasing" to infinity for a finite time.

The paper develops a general idea on the polystability of motion of nonlinear systems (see [1, 6]).

2. THE GENERAL PROBLEM ON POLYSTABILITY

According to [6] we consider a system of perturbed motion equations

$$dx_i/dt = f_i(t, x_1, \dots, x_s), \qquad x_i(t_0) = x_{i0}, \tag{2.1}$$

where $x_i \in R^{n_i}$, $t \in \mathcal{T}_\tau$, $\mathcal{T}_\tau = [\tau, +\infty)$, $\tau \in R$, $t_0 \in \mathcal{T}_i$, $\mathcal{T}_i \subseteq R$, $f_i: \mathcal{T}_\tau \times R^{n_1} \times R^{n_2} \times \cdots \times R^{n_s} \to R^{n_i}$, and it is assumed that $f_i(t, x_1, \dots, x_s) = 0$ for all $t \in \mathcal{T}_\tau$ if and only if $x_1 = x_2 = \cdots = x_s = 0$. Together with (2.1), we shall use the vector notion of the system

$$S: dx/dt = f(t, x), \qquad x(t_0) = x_0, \tag{2.2}$$

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where $x \in R^n$, $n = \sum_{i=1}^{s} n_i$, $f: \mathcal{T}_\tau \times R^n \to R^n$, $x_0 = [x_{10}^T, \cdots x_{s0}^T]^T$. It is clear that $f(t, x) = 0$ for all $t \in \mathcal{T}_\tau$ if and only if $x = 0$.

DEFINITION 2.1. System (2.2) is called polystable (on $\mathcal{T}_\tau$), if and only if its zero solution $(x = 0) \in R^n$ is stable in some type (on $\mathcal{T}_\tau$) and attractive (on $\mathcal{T}_\tau$) with respect to groups of variables $[x_i^T]$, $i = 1, 2, \ldots, s$ (with respect to the totality of the groups of variables $[x_1^T, \ldots, x_l^T]$, $l < s$).

The expression "on $\mathcal{T}_\tau$" in Definition 2.1 and below is omitted iff $\mathcal{T}_\tau = R$.

Remark 2.1. When the polystability of solution $x = 0$ of (2.2) is discussed with respect to all groups of variables $[x_i^T]$, $i = 1, 2, \ldots, s$, system (2.2) is defined in the domain

$$B(p) = \{x : \|[x_1^T, \ldots, x_s^T]^T\| < p\}, \qquad p = \text{const} > 0,$$

or in R^n as usual.

Remark 2.2. If the polystability of solution $x = 0$ of (2.2) is discussed with respect to a totality of groups of variables $[x_1^T, \ldots, x_l^T]$, $l < s$, then it is sufficient to define system (2.2) in the domain

$$B_l(p) = \{x_i^T : \|[x_1^T, \ldots, x_l^T]^T\| < p^*\}, \qquad p^* = \text{const} > 0,$$
$$D_l = \{x_k^T : 0 < \|[x_{l+1}^T, \ldots, x_s^T]^T\| < +\infty\}, \qquad k = l+1, \ldots, s.$$

Here the solution $x(t; t_0, x_0) = [x_1^T(t), \ldots, x_s^T(t)]^T$ of system (2.2) is assumed to be continuous along $[x_{l+1}^T, \ldots, x_s^T]$; i.e., the solution $x(t; t_0, x_0)$ is definite for all $t \in \mathcal{T}_\tau$ for which $\|[x_1^T(t), \ldots, x_l^T(t)]^T\| < p^*$.

Remark 2.3. The construction of sufficient polystability conditions for system (2.2) assume Definition 2.1 to be concretized as to the dynamical properties of groups of variables $[x_i^T]$, $i \in [1, s]$.

Remark 2.4. All the definitions on stability (on $\mathcal{T}_\tau$) with respect to $\mathcal{T}_i$ used in this paper are found in [2].

Thus the construction of sufficient (and necessary) conditions ensuring polystability of the zero solution of (2.2) in terms of Definition 2.2 makes up the general problem on the polystability of motion.

3. THE LYAPUNOV FUNCTION FOR POLYSTABILITY INVESTIGATION

In order to apply the method of the Lyapunov function [5] to the problem in question, we introduce classes of functions with particular properties.

DEFINITION 3.1. The function $V: R \times R^n \to R$ is

(i) positive semi-definite on $\mathcal{T}_\tau$, $\tau \in R$, iff there is a time-invariant connected neighbourhood $\mathcal{N}$ of $x = 0$, such that

(a) V is continuous in $(t, x) \in \mathcal{T}_\tau \times \mathcal{N}$;

(b) V is a non-negative on $\mathcal{N}$: $V(t, x) \geq 0$, $\forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N}$;

(c) V vanishes at the origin : $V(t, 0) = 0$, $\forall t \in \mathcal{T}_\tau$;

(ii) positive semi-definite on $\mathcal{T}_\tau \times \mathcal{S}$ iff (i) holds for $\mathcal{N} = \mathcal{S}$.

DEFINITION 3.2. Function $V: R \times R^n \to R$ is:

(i) positive definite on $\mathcal{T}_\tau$, $\tau \in R$, with respect to variables $[x_1^T, \ldots, x_l^T] \in R^{n_1 + \cdots + n_l}$ iff there is a time-invariant connected neighbourhood $\mathcal{N}_l$ of $x_1 = \cdots = x_l = 0$, $\mathcal{N}_l \subseteq R^{n^*}$, such that both are positive semi-definite on $\mathcal{T}_\tau \times \mathcal{N}$ and there exists a positive definite function w on $\mathcal{N}_l$, obeying

$$w(x_1^T, \ldots, x_l^T) \leq V(t, x) \, \forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N}_l \times D_l;$$

(ii) positive definite on $\mathcal{T}_\tau \times \mathcal{S}^*$ with respect to variables $[x_1^T, \ldots, x_l^T]$, $l < s$, iff (i) holds for $\mathcal{N}_l = \mathcal{S}^*$.

The expression "on $\mathcal{T}_\tau$" is omitted iff all corresponding requirements hold for every $\tau \in R$.

DEFINITION 3.3. Function $V: R \times R^n \to R$ is:

(i) decreasing on $\mathcal{T}_\tau$, $\tau \in R$, with respect to variables $[x_1^T, \ldots, x_l^T]$, $l < s$, iff there is a time-invariant neighbourhood $\mathcal{N}_l$ of $x_1 = \cdots x_l = 0$ and a positive definite function u on $\mathcal{N}_l$, $u: R^{n^*} \to R$ such that $V(t, x) \leq u(x) \, \forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N}_l$;

(ii) decreasing on $\mathcal{T}_\tau \times \mathcal{S}^*$ with respect to variables $[x_1^T, \ldots, x_l^T]$, $l < s$, iff (i) holds for $\mathcal{N}_l = \mathcal{S}^*$.

4. MAIN RESULT

For any function

$$V \in C(\mathcal{T}_\tau \times R^n, R_+) \tag{4.1}$$

we define the function

$$D^+ V(t, x) = \lim_{\theta \to 0^+} \sup[V(t + \theta, x + \theta f(t, x)) - V(t, x)]\theta^{-1}. \tag{4.2}$$

Suppose that $V(t, x)$ and system (2.2) are defined in the domain

$$\mathcal{T}_\tau \times B_1(p) \times B_2(p), \qquad p = \text{const} > 0, \tag{4.3}$$

and the following stability definition holds true for it.

DEFINITION 4.1. System (2.2) is called polystable (on $\mathcal{T}_\tau$) if its zero solution $[x_1^T, x_2^T]^T = 0$ is

(i) uniformly $[x_1^T, x_2^T]$-stable with respect to $\mathcal{T}_i$;

(ii) uniformly asymptotically x_2^T-stable with respect to $\mathcal{T}_i$.

THEOREM 4.1. *Let the vector function* $f = [f_1^T, f_2^T]^T$ *be continuous on* $R_+ \times B_1(p) \times B_2(p)$ *(on* $\mathcal{T}_\tau \times B_1(p) \times B_2(p)$*). If there exists*

(1) *an open connected time-invariant neighbourhood* $\mathcal{S}$ *of* $x = 0$;

(2) *a function* $V(t, x)$ *is*

 (i) *positive definite on* $\mathcal{S}$ *(on* $\mathcal{T}_\tau \times \mathcal{S}$*)*;

 (ii) *decreasing on* $\mathcal{S}$ *(on* $\mathcal{T}_\tau \times \mathcal{S}$*)*;

(3) *a function* $D^+V(t, x)$ *is*

 (i) *negative semi-definite on* $R \times \mathcal{S}$ *(on* $\mathcal{T}_\tau \times \mathcal{S}$*)*;

 (ii) x_2^T*-negative definite on* $R \times \mathcal{S}$ *(on* $\mathcal{T}_\tau \times \mathcal{S}$*)*.

Then system (2.2) *is polystability (on* $\mathcal{T}_\tau$*) in the sense of Definition* 4.1.

Proof. If conditions (1), (2), and (3i) hold for system (2.2) and function $V(t, x)$, then all the hypotheses of Theorem 5 from [2] are fulfilled and state that $(x = 0) \in R^N$, $N = n_1 + n_2$, is uniformly stable (on $\mathcal{T}_\tau$). If conditions (1), (2), and (3ii) hold for (2.2) and function (4.2), then all the hypotheses of Theorem 7 from [2] are fulfilled and state that $(x = 0) \in R^N$ is uniformly asymptotically x_1^T-stable (on $\mathcal{T}_\tau$). The theorem is proved.

Further we suppose that the polystability of (2.2) for $s = 2$ is investigated in the domain

$$\mathcal{T}_\tau \times B_1(p) \times D_2, D_2 = \{x_2 : 0 < \|x_2\| < \infty\}. \tag{4.4}$$

We formulate a theorem generalizing some results from [8].

THEOREM 4.2. *Let the vector function* $f = [f_1^T, f_2^T]^T$ *in* (2.2) *be continuous on* $R \times B_1(p) \times D_2$ *(on* $\mathcal{T}_\tau \times B_1(p) \times D_2$*). If there exists*

(1) *an open connected time-invariant neighbourhood* $\mathcal{S}^*$ *of* $(x = 0) \in R^{n_1}$;

(2) *a function* $V(t, x)$ *that is*

 (i) x_1^T*-positive definite on* $\mathcal{S}^*$ *(on* $\mathcal{T}_\tau \times \mathcal{S}^*$*)*;

(ii) *decreasing on* $\mathcal{S}^*$ (*on* $\mathcal{T}_\tau \times \mathcal{S}^*$);

(iii) x_1^T-*decreasing on* $\mathcal{S}^*$ (*on* $\mathcal{T}_\tau \times \mathcal{S}^*$);

(3) *a function* $D^+V(t, x)$ *that is*

(i) *negative semi-definite on* $R \times \mathcal{S}^*$ (*on* $\mathcal{T}_\tau \times \mathcal{S}^*$);

(ii) x_1^T-*negative definite on* $R \times \mathcal{S}^*$ (*on* $\mathcal{T}_\tau \times \mathcal{S}^*$);

(iii) *negative definite on* $R \times \mathcal{S}$ (*on* $\mathcal{T}_\tau \times \mathcal{S}^*$),

then, respectively,

(a) *conditions* (1), (2i), and (3i) *are sufficient for* x_1^T-*stability of state* $(x = 0) \in R^n$ *of* (2.2) (*on* $\mathcal{T}_\tau$);

(b) *conditions* (1), (2i), (2ii), *and* (3i) *are sufficient for uniform* x_1^T-*stability of state* $(x = 0) \in R^n$ *of* (2.2) (*on* $\mathcal{T}_\tau$);

(c) *conditions* (1), (2i), (2ii), and (3ii) are sufficient for asymptotic x_1^T-*stability of state* $(x = 0) \in R^n$ *of* (2.2) (*on* $\mathcal{T}_\tau$).

Proof. To prove Theorem 4.2 we consider function (4.1) which is locally Lipshitzian on $R \times \mathcal{S}^*$ (on $\mathcal{T}_\tau \times \mathcal{S}^*$) and its upper right-hand side Dini derivative (4.2) on $R \times \mathcal{S}$ (on $\mathcal{T}_\tau \times \mathcal{S}$). Let conditions (1), (2), and (3a) of Theorem 4.2 be fulfilled. Then there is a function $a \in K$ such that $a(\|x\|) \leq V(t, x)$ on $\mathcal{T}_\tau \times B_1(p) \times D_2$. Because of (2i) we have $D^+V(t, x) \leq 0$ on $R \times \mathcal{S}$ (on $\mathcal{T}_\tau \times \mathcal{S}$). Therefore all hypotheses of Theorem 5.1 from [8] hold. This proves the x_1^T-stability of state $(x = 0) \in R^N$ of (2.2) (on $\mathcal{T}_\tau$). Statements (b) and (c) of Theorem 4.2 can be proved similarly.

5. Application

We consider the two-component differential system

$$dx_i/dt = f_i(t, x_i) + g_i(t, x_1, x_2), \qquad i = 1, 2, \tag{5.1}$$

where $x_i \in R^{n_i}$ and $f_i \in C(\mathcal{T}_\tau \times R^{n_i}, R^{n_i})$, $g_i \in C(\mathcal{T}_\tau \times R^{n_1} \times R^{n_2}, R^{n_i})$. Let $f_i(t, 0) = g_i(t, 0, 0) = 0$ for all $t \in \mathcal{T}_\tau$.

Definition 5.1. System (5.1) is called exponentially polystable (on $\mathcal{T}_\tau$) if in a neighbourhood $\mathcal{N} \subseteq R^N$ containing the point $x = 0$ the estimate

$$\|x_1(t; t_0, x_0)\|^{r_1} + \|x_2(t; t_0, x_0)\|^{r_2} \leq a\|x_0\| \exp(-\lambda(t - t_0))$$

holds, where $a > 0$, $\lambda > 0$, $t \geq t_0$, $r_1 \neq r_2$ are positive integers. The constants a and λ may depend on $\mathcal{N}$.

DEFINITION 5.2. (See [9]). Two functions φ_l, φ_2 are said to be of the same order of magnitude if there exist positive constants α_i, β_i, $i = 1, 2$, such that

$$\alpha_i \varphi_i(r) \leq \varphi_j(r) \leq \beta_i \varphi_i(r), \qquad i \neq j; i, j = 1, 2.$$

Let us prove the following result.

THEOREM 5.1. *Let the vector function $f = [f_1^T, f_2^T]^T$ in (5.1) be continuous on $\mathcal{T}_\tau \times B_1(p) \times B_2(p)$. If there exists*

(i) *an open connected time-invariant neighbourhood $\mathcal{N}$ of $(x = 0) \in R^N$;*

(ii) *the function $V(t, x_1, x_2)$ and a constant $A > 0$ such that*

(a) $A(\|x_1\|^{r_1} + \|x_2\|^{r_2}) \leq V(t, x_1, x_2)$;

(b) *they are decreasing on $\mathcal{S}$ (on $\mathcal{T}_\tau \times \mathcal{S}$); i.e. $V(t, x_1, x_2) \leq \varphi_1(\|x\|)$, where $\|x\| = \|x_1\| + \|x_2\|$, $\varphi_1 \in K$;*

(iii) *the function $D^+V(t, x_1, x_2)$ such that $[D^+V(t, x_1, x_2)]_{f+g} \leq -\varphi_2(\|x\|)$ on $\mathcal{T}_\tau \times \mathcal{S}$, where $\varphi_2 \in K$;*

(iv) *functions φ_1, φ_2 are of the same order of magnitude.*

Then the system (5.1) is exponentially polystable.

Proof. According to condition (iv) of Theorem 5.1 there exist constants α_1 and β_1 such that

$$\alpha_1 \varphi_1(r) \leq \varphi_2(r) \leq \beta_1 \varphi_1(r). \tag{5.2}$$

From condition (ii)(b) and (iii) of Theorem 5.1 we have

$$[D^+V(t, x_1, x_2)]_{f+g} \leq -\alpha_1 V(t, x_1, x_2)$$

or

$$V(t, x_1(t), x_2(t)) \leq V(t_0, x_{10}, x_{20}) \exp[-\alpha_1(t - t_0)], \qquad t \geq t_0.$$

In view of condition (ii)(a) we get

$$\|x_1(t; t_0, x_0)\|^{r_1} + \|x_2(t; t_0, x_0)\|^{r_2} \leq A^{-1}\varphi_1(\|x_0\|) \exp[-\alpha_1(t - t_0)].$$

The theorem is proved.

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